ECE 443/643 Test 2

November 9, 2011

1. Let $x(t) = (A + m(t))\cos(2\pi f_c t)$ be the current (in amps) delivered to the 50Ω antenna of a radio station. The station operates with $f_c = 790$kHz and a carrier power of 5 kW. Assume $f_{LO} > f_c$

   (a) *(5)* Find A.

   (b) *(5)* If 75% modulation is used, what is the PEP?

   (c) *(10)* Suppose a superhet with an intermediate frequency of 455 kHz is used to receive the signal. What is the image frequency?

2. Let $x(t) = \Re(m_-(t)e^{2\pi j f_c t})$, where $m_-(t) = m(t) - j\hat{m}(t)$, $m(t) = (\text{sinc}(t))^2$, $f_c = 10$, and $\hat{m}(t)$ is the Hilbert transform of m.

   (a) *(10)* Sketch $M(f)$, $M_-(f)$, and $X(f)$.

   (b) *(15)* Sketch a block diagram of a modulator that generates $x(t)$ from $m(t)$.

   (c) *(5)* What kind of modulation is this?

3. Let $x(t) = \Pi(t)$.

   (a) *(15)* Find $\hat{x}$, the Hilbert transform of x. Assume $\int_{-T}^{T} \frac{1}{\pi\lambda}\, d\lambda = 0$

   (b) *(5)* Sketch $\hat{x}$.

   (c) *(5)* Is $\hat{x}$ a physical signal? In otherwords, is it finite, real, etc. Why? If it is not, suggest some ways to make it physical.

   (d) *(10)* Find the energy spectral density of $\hat{x}$.

Test 2

1.a

Carrier signal $= A\cos(2\pi f_c t)$

Carrier power $= P_c = \underbrace{\frac{A^2}{2}}_{i_{RMS}^2} R = 5k \rightarrow A = \sqrt{\frac{2P_c}{R}} = \boxed{14.14}$ (amps)

1.b

75% modulation $\rightarrow \mu = \frac{m_p}{A} = 0.75 \rightarrow m_p = \frac{3}{4}A$

$$PEP = \frac{(A+m_p)^2}{2}R = \frac{\left(\frac{7}{4}A\right)^2}{2}R = \boxed{15.31\ kW}$$

1.c

$f_{LO} > f_c \rightarrow f_{LO} = f_c + f_{IF}$

$f_{IM} = f_c + 2f_{IF} = \boxed{1700\ kHz}$

If you forgot the formula, you could remember that f_{IM} is mixed down to f_{IF}.

$$f_{IF} = f_{IM} \pm f_{LO} \rightarrow f_{IM} = f_{IF} \mp f_{LO} = f_{IF} \mp (f_c + f_{IF}) = \begin{cases} -f_c & \text{(desired)} \\ f_c + 2f_{IF} & \text{(image)} \end{cases}$$

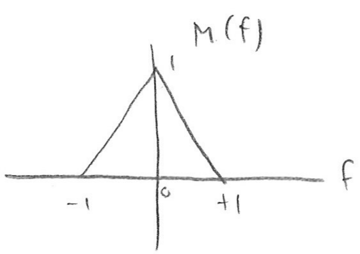
2.a

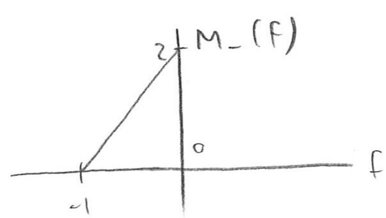
$m(t) = \text{sinc}^2(t) \xrightarrow{\mathcal{F}} M(f) = \Lambda(f)$

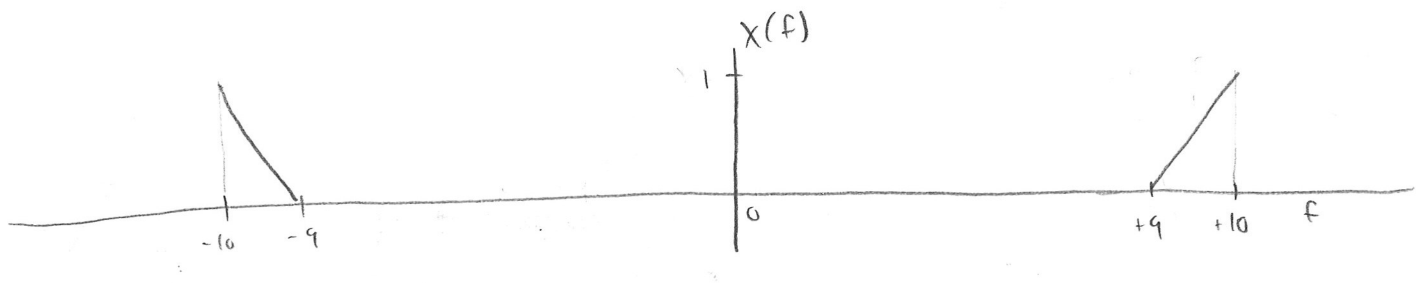
$m_-(t) = m - j\hat{m} \xrightarrow{\mathcal{F}} M - j\hat{M} = M - j(-j\,\text{sgn}(f))M = (1-\text{sgn}(f))M(f) = 2u(-f)M(f)$

$x(t) = \text{Re}(m_-(t)e^{2\pi j f_c t}) = \frac{1}{2}m_-(t)e^{2\pi j f_c t} + \frac{1}{2}m_-^*(t)e^{-2\pi j f_c t}$

$\xrightarrow{\mathcal{F}} X(f) = \frac{1}{2}M_-(f-f_c) + \frac{1}{2}M_-^*(-f+f_c)$







Test 2

2.b)

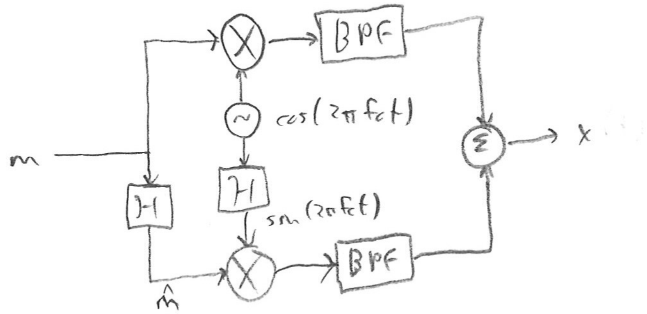
$$x(t) = \mathrm{Re}\left(m_- e^{2\pi j f_c t}\right) = \mathrm{Re}\left((m - j\hat{m}) e^{2\pi j f_c t}\right)$$

$$= \mathrm{Re}\left(m \cos(\omega_c t) + \hat{m} \sin(\omega_c t) + j m \sin(\omega_c t) - j \hat{m} \cos(\omega_c t)\right)$$

we know that m and $\hat{m}$ are real.

$$x(t) = m(t) \cos(2\pi f_c t) + \hat{m}(t) \sin(\omega_c t)$$

It's pretty easy to just draw the diagram from the expression for x. We just have to remember that $m \rightarrow \boxed{\mathcal{H}} \rightarrow \hat{m}$, $\cos \rightarrow \boxed{\mathcal{H}} \rightarrow \sin$, and mixers should be followed by filters (BPF in this case).



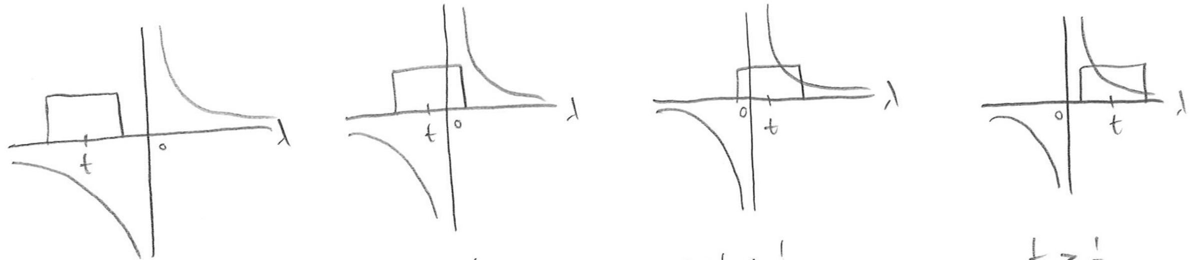
2.c)

lower sideband amplitude modulation

3.a)

$$\hat{x}(t) = \int_{-\infty}^{\infty} \Pi(t-\lambda)\frac{1}{\pi\lambda}\,d\lambda \qquad \text{Assume } \int_{-T}^{T}\frac{d\lambda}{\pi\lambda} = 0$$

Consider 4 regions:



$t < -\frac{1}{2}$ $\quad -\frac{1}{2} < t < 0$ $\quad 0 < t < \frac{1}{2}$ $\quad t > \frac{1}{2}$

The $t < -\frac{1}{2}$ and $t > \frac{1}{2}$ cases are essentially the same. They don't involve the singularity.

$$\int_{-\infty}^{\infty}\frac{1}{\pi\lambda}\Pi(t-\lambda)\,d\lambda = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}}\frac{d\lambda}{\pi\lambda} = \frac{1}{\pi}\ln|\lambda|\Big]_{\lambda=t-\frac{1}{2}}^{t+\frac{1}{2}} = \frac{1}{\pi}\ln\left|t+\tfrac{1}{2}\right| - \frac{1}{\pi}\ln\left|t-\tfrac{1}{2}\right|$$

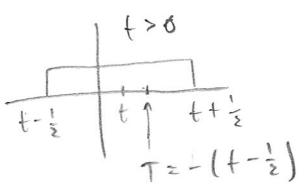
$$= \frac{1}{\pi}\ln\left|\frac{t+\frac{1}{2}}{t-\frac{1}{2}}\right|, \quad |t| > \tfrac{1}{2}$$

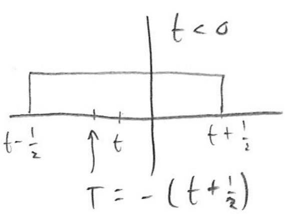
The $|t| < \frac{1}{2}$ regions have the singularity. Split the integral into a symmetric part and a remainder.

$$t>0: \int_{t-\frac{1}{2}}^{t+\frac{1}{2}}\frac{1}{\pi\lambda}\,d\lambda = \int_{-T}^{T}\frac{d\lambda}{\pi\lambda} + \int_{T}^{t+\frac{1}{2}}\frac{d\lambda}{\pi\lambda} = 0 + \frac{1}{\pi}\ln|\lambda|\Big]_{\lambda=T}^{t+\frac{1}{2}} = \frac{1}{\pi}\ln\left|\frac{t+\frac{1}{2}}{T}\right|$$

$$t<0: \int_{t-\frac{1}{2}}^{t+\frac{1}{2}}\frac{1}{\pi\lambda}\,d\lambda = \int_{t-\frac{1}{2}}^{-T}\frac{d\lambda}{\pi\lambda} + \int_{-T}^{T}\frac{d\lambda}{\pi\lambda} = \frac{1}{\pi}\ln|\lambda|\Big]_{\lambda=t-\frac{1}{2}}^{-T} + 0 = \frac{1}{\pi}\ln\left|\frac{-T}{t-\frac{1}{2}}\right|$$

What is T? Whatever the edge of the box closest to 0:





$$T = \begin{cases} -(t+\frac{1}{2}) & t<0 \\ -(t-\frac{1}{2}) & t>0 \end{cases}$$

Putting it all together: $\hat{x}(t) = \boxed{\frac{1}{\pi}\ln\left|\frac{t+\frac{1}{2}}{t-\frac{1}{2}}\right|}$

test 2

3.a (alternate)

A short cut, which is an abuse and may not give correct answers in other contexts, is to use $\int_{-\infty}^{t} \frac{d\lambda}{\pi\lambda} = \ln|t|$. This works out for us, because we have defined $\int_{-T}^{T} \frac{d\lambda}{\pi\lambda} = 0$. For any limits a & b, with $a<0<b$, we can do:

$$\int_a^b \frac{d\lambda}{\pi\lambda} = \int_a^{-T} \frac{d\lambda}{\pi\lambda} + \int_{-T}^{T} \frac{d\lambda}{\pi\lambda} + \int_T^b \frac{d\lambda}{\pi\lambda} = \int_a^{-T} \frac{d\lambda}{\pi\lambda} + \int_T^b \frac{d\lambda}{\pi\lambda}$$

$$= \frac{1}{\pi}\left(\ln|-T| - \ln|a|\right) + \frac{1}{\pi}\left(\ln|T| - \ln|-T|\right) + \frac{1}{\pi}\left(\ln|b| - \ln|T|\right)$$

$$= \frac{1}{\pi}\left(\ln|b| - \ln|a|\right)$$

everything cancels because of our definition, so it's ok.

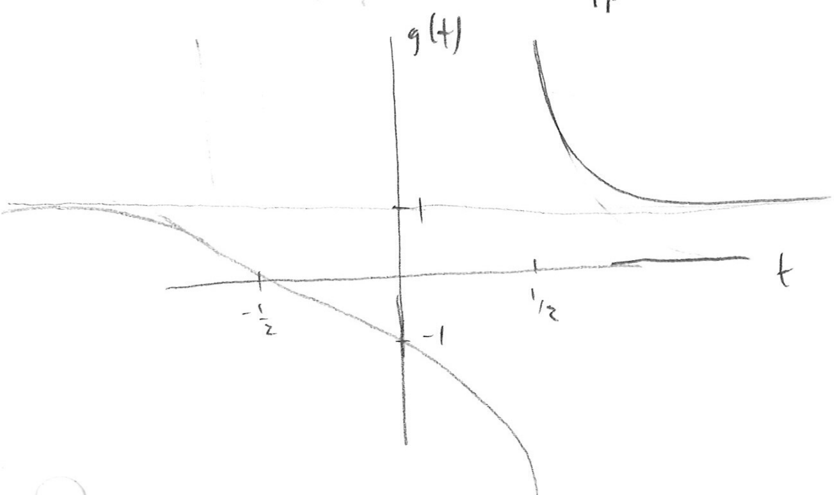
Fortunately, when $a<b<0$ and $0<a<b$, everything works out the same.

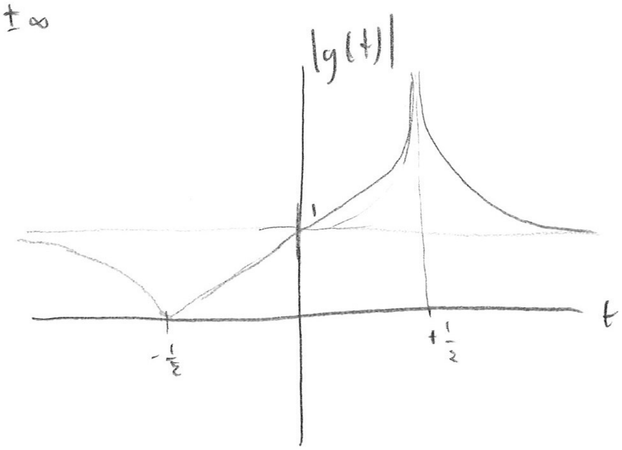
$$\hat{x}(t) = \int_{-\infty}^{\infty} \Pi(t-\lambda)\frac{1}{\pi\lambda}\,d\lambda = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \frac{1}{\pi\lambda}\,d\lambda = \frac{1}{\pi}\ln|\lambda|\Big]_{\lambda=t-\frac{1}{2}}^{t+\frac{1}{2}} = \boxed{\frac{1}{\pi}\ln\left|\frac{t+\frac{1}{2}}{t-\frac{1}{2}}\right|}$$

3.b

You probably wish you had a graphing calculator. It's not necessary, though.

Let $g(t) = \frac{t+\frac{1}{2}}{t-\frac{1}{2}}$. g has a zero at $t=-\frac{1}{2}$, a pole at $t=\frac{1}{2}$, and asymptotically approaches 1 at $\pm\infty$

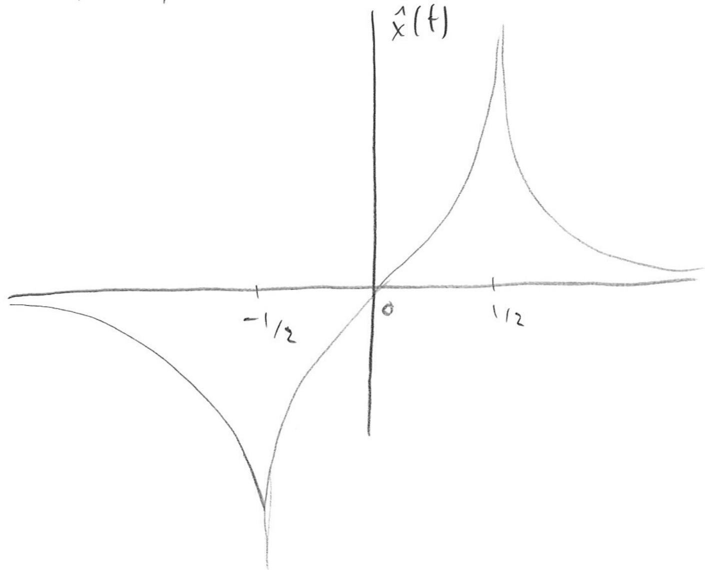




test2

3.b cont.

The interesting points are where $|g|$ goes to 1, 0, and ∞, because these are the interesting arguments of $\ln(\cdot)$. $\ln(\infty) = \infty$, $\ln(0) = -\infty$, and $\ln(1) = 0$



3.c

The output goes to infinity, so it's obviously not physical. It's also not causal (technically, our input wasn't either, but clearly the output starts before the input). We can solve the causality problem by shifting the impulse response "enough" so that the energy contributed by $h(t), t<0$ is sufficiently small, and then multiply $h(t)$ by $u(t)$. $h(t) \rightarrow h(t-\tau)u(t)$

The infinities are caused by the high frequencies present in the input. If we low-pass filter the input first, the output will be finite.

3.d

As we found in the homework, the ESD of the Hilbert transform is the ESD of the original. If you forgot this, then:

$$ESD\{\hat{x}(t)\} = |\hat{X}(f)|^2 = |-j\,sgn(f)\,X(f)|^2 = |-j\,sgn(f)|^2 |X(f)|^2 = 1 \cdot |X(f)|^2$$

$$X(f) = sinc(f) \rightarrow ESD\{\hat{x}\} = \boxed{sinc^2(f)}$$